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An Extension of the Markov Inequality

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1. INTRODUCTION

Denote by π_n the set of algebraic polynomials of degree not exceeding n. Set

$$\|f\|_C := \max_{-1 \leqslant x \leqslant 1} |f(x)|.$$

The inequality

$$\|f'\|_C \leqslant n^2 \|f\|_C \qquad (f \in \pi_n), \tag{1}$$

is a well-known classical result in approximation theory (see Rivlin [1]); it was proved by A. A. Markov in 1889. Sometimes (1) is written in the form

$$\|f'\|_C \leqslant \|T_n'\|_C \, \|f\|_C, \tag{2}$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind, i.e.,

$$T_n(x) = \cos(n \arccos x).$$

Let us also note the evident fact that

$$V(f; [-1, 1]) \leqslant V(T_n; [-1, 1]) \|f\|_C \qquad (f \in \pi_n), \tag{3}$$

where $V(f; [-1, 1])$ denotes the total variation of f in $[-1, 1]$. Using the notation

$$\|f\|_p := \left\{ \int_{-1}^{1} |f(x)|^p \, dx \right\}^{1/p} \qquad \text{for} \quad 1 \leqslant p < \infty,$$

one can rewrite (3) as

$$\|f'\|_1 \leqslant \|T_n'\|_1 \, \|f\|_C.$$

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Here, as in (2), the equality is attained if and only if $f = \pm T_n$. So, the famous Chebyshev polynomials T_n have a maximal L_1 and C norm for its first derivative in the set $\{f \in \pi_n : \|f\|_C \leqslant 1\}$. Whether T_n preserves its extremal role in the corresponding L_p-problem

$$\|f'\|_p \to \sup; \qquad f \in \pi_n, \|f\|_C \leqslant 1,$$

for $1 < p < \infty$, is the central question discussed in the present paper. We give here an affirmative answer to this question showing that

$$\|f'\|_p \leqslant \|T_n'\|_p \, \|f\|_C \qquad (f \in \pi_n),$$

for each $p \in (1, \infty)$.

2. Auxiliary Lemmas

We prove in this section three propositions which will be needed for the proof of the main result.

LEMMA 1. *Let $\tau(t)$ be an arbitrary trigonometric polynomial of order n with a uniform norm equal to 1. Suppose that α is a real number from the interval $(-1, 1)$. Denote by ξ the point from $(0, \pi/n)$ for which $\cos n\xi = \alpha$. Let η be an arbitrary point from $(-\infty, \infty)$ for which $\tau(\eta) = \alpha$. Then*

$$n \sin n\xi \geqslant |\tau'(\eta)|. \tag{4}$$

The equality is attained if and only if $\tau(t) = \cos nt$ (up to translation and multiplication by -1).

Proof. This seems to be a known fact. An anologous statement was used, for instance, by Taikov [2]. For the sake of completeness we sketch here the proof.

The inequality is obvious in the case $\tau'(\eta) = 0$. Suppose that $\tau'(\eta) \neq 0$. Let us assume that (4) does not hold for some τ and η. Then the function

$$g(t) = \varepsilon_1 \tau(t - \xi + \eta) - \cos nt, \qquad \varepsilon_1 = -\text{sign}\, \tau'(\eta),$$

would have three zeros in $[0, \pi/n]$ and (because of the oscillating property of $\cos nt$) another $2n - 1$ zeros in $[-\pi, 0] \cup [\pi/n, \pi)$. But $g(t)$ is a trigonometric polynomial of order n, thus it has at most $2n$ zeros in $[-\pi, \pi)$. The contradiction proves the lemma.

We present in the sequel an analogy of Lemma 1 in the algebraic case.

Let $\{\theta_k\}_0^n$ be the extremal points of $T_n(x)$ in $[-1, 1]$. It is known (see Rivlin [1]) that $\theta_0 = -1$, $\theta_n = 1$ and

$$T_n(\theta_k) = (-1)^{n-k}, \qquad k = 0,..., n.$$

Denote by Ω_n the set of those polynomials f from the class $\{g \in \pi_n, \|g\|_C = 1\}$ which possess $(m+1)$ points of alternation in $[-1, 1]$ $(m = 1,..., n)$, i.e., for which there exist $m+1$ points $\{x_i\}_0^m$, $-1 = x_0 < \cdots < x_m = 1$, such that

$$f(x_k) = (-1)^{m-k}, \qquad k = 0,..., m,$$

and $f(x)$ is a monotone function in $[x_k, x_{k+1}]$, $k = 0,..., m-1$. Suppose that $f \in \Omega_n$. Evidently there is an $i \in \{0,..., m-1\}$ such that $x_i < 0 \leqslant x_{i+1}$. Consider the partition of $[-1, 1]$ into subintervals $[x_0, x_1],..., [x_i, 0]$, $[0, x_{i+1}],..., [x_{m-1}, x_m]$ which we denote, for simplicity, by $I_0,..., I_m$, respectively. Introduce the points t_1 and t_2 defined by the conditions

$$\begin{aligned} &t_1 \in [\theta_i, \theta_{i+1}], \qquad && T_n(t_1) = f(0), \\ &t_2 \in [\theta_{i+n-m}, \theta_{i+n-m+1}], \qquad && T_n(t_2) = f(0). \end{aligned}$$

Denote the intervals

$$[\theta_0, \theta_1],..., [\theta_i, t_1], [t_2, \theta_{i+n-m+1}],..., [\theta_{n-1}, \theta_n]$$

by $I_0^*,..., I_m^*$, respectively. We shall refer frequently to the correspondence between I_k and I_k^*, $k = 0,..., m$.

LEMMA 2. *Suppose that $f \in \Omega_n$, $\alpha \in (-1, 1)$ and $k \in \{0,..., m\}$. Let the points ξ and η be defined by the conditions*

$$\begin{aligned} &\xi \in I_k^*, \qquad && T_n(\xi) = \alpha, \\ &\eta \in I_k, \qquad && f(\eta) = \alpha. \end{aligned}$$

Then

$$|T_n'(\xi)| \geqslant |f'(\eta)| \tag{5}$$

and the equality is attained if and only if $f = T_n$.

Proof. Suppose that f has $m+1$ points of alternation. If $m = n$ then $f = T_n$ and (5) holds. We assume in what follows that $f \neq T_n$. Clearly $\eta \neq \pm 1$ since $|f(\pm 1)| = 1 > |\alpha|$. Suppose that $0 \leqslant \eta < 1$. Let the intervals $I = [z_1, z_2]$ and $I^* = [z_1^*, z_2^*]$ be corresponding and $I \subset [0, 1]$. We shall show that

$$z_1 < z_1^*, \qquad z_2 \leqslant z_2^*. \tag{6}$$

Moreover, the equality holds in the case $z_2 = z_2^* = 1$ only. We apply an induction. If $I = I_m$ then $z_2 = z_2^* = 1$ and clearly $z_1 < z_1^*$ since the assumption $z_1^* \leqslant z_1$ implies that $f(x) - T_n(x)$ has more than n zeros in $[-1, 1]$. Suppose that $I = I_k$, $k < m$. Let us assume that the relations (6) hold for $I = I_{k+1}$. Then $z_2 < z_2^*$ since z_2 is a first end point of I_{k+1}. Suppose that $z_1^* \leqslant z_1$. Then the polynomial $f(x) - T_n(x)$ would have two zeros at least in $[z_1^*, z_2^*]$ and $n-1$ other zeros in $[-1, z_1^*] \cup [z_2^*, 1]$, i.e., more than n. Therefore $z_1 < z_1^*$. The assertion (6) is proven.

Now we shall show that $\eta < \xi$. Suppose that $\xi \in I^* = [z_1^*, z_2^*]$ and $\eta \in I = [z_1, z_2]$. Let us assume that $\xi \leqslant \eta$. Since $z_1 < z_1^*$ and $z_2 \leqslant z_2^*$, the polynomial $f(x) - T_n(x)$ will have at least two zeros in (z_1^*, z_2^*) and $n-1$ zeros in $[-1, z_1^*] \cup [z_2^*, 1]$, a contradiction. Therefore

$$0 \leqslant \eta < \xi < 1. \tag{7}$$

Consider the trigonometric polynomials

$$T_n(\cos t) = \cos nt,$$
$$\tau(t) = f(\cos t).$$

It follows from the evident identities

$$T_n(x) = \cos(n \arccos x),$$
$$f(x) = \tau(\arccos x)$$

that

$$T_n'(\xi) = n \sin(n \arccos \xi) \cdot (1 - \xi^2)^{-1/2}, \tag{8}$$

$$\tau'(\eta) = -\tau'(\arccos \eta) \cdot (1 - \eta^2)^{-1/2}. \tag{9}$$

But (7) implies

$$(1 - \eta^2)^{-1/2} < (1 - \xi^2)^{-1/2}. \tag{10}$$

On the other hand, according to Lemma 1,

$$|n \sin(n \arccos \xi)| > |\tau'(\arccos \eta)| \tag{11}$$

since

$$\cos(n \arccos \xi) = T_n(\xi) = \alpha,$$
$$\tau(\arccos \eta) = f(\eta) = \alpha.$$

The assertion of the lemma follows from Eqs. (8)–(11).

The proof is completely similar in the case $-1 < \eta \leqslant 0$. The lemma is proved.

Remark 1. The requirement that $f(x)$ is monotone between the points of alternation is not essential. Lemma 2 can be proved in the same fashion without this requirement, assuming that η is an arbitrary point from I_k for which $f(\eta) = \alpha$.

LEMMA 3. *Suppose that $F(x)$ is a convex, increasing function on $[0, \infty)$ and $F(0) = 0$. Then*

$$\int_{-1}^{1} F(|f'(x)|)\, dx \leqslant \int_{-1}^{1} F(|T_n'(x)|)\, dx \tag{12}$$

for each $f \in \Omega_n$. The equality is attained if and only if $f = T_n$.

Proof. We follow the idea used by Taikov [2] in the solution of an analogous problem for trigonometric polynomials.

There is an $M > 0$ such that $\|f'\|_C \leqslant M \|f\|_C$ for each $f \in \pi_n$. With every $\alpha \in [0, M]$ we associate the function

$$\varphi_\alpha(x) := \begin{cases} 0, & 0 \leqslant x < \alpha, \\ x, & \alpha \leqslant x \leqslant M. \end{cases}$$

Divide the interval $[0, M]$ into N equal parts by the points $\alpha_\kappa = (k/N) \cdot M$, $k = 0, ..., N$. Next we construct the function

$$\Phi_N(x) = \sum_{k=1}^{N-1} \beta_k \varphi_{\alpha_k}(x)$$

to satisfy the interpolation conditions

$$\Phi_N(\alpha_k) = F(\alpha_k), \qquad k = 1, ..., N-1.$$

Since F is convex and $F(0) = 0$, we conclude that $\beta_k > 0$, $k = 1, ..., N-1$. Evidently the functions $\Phi_N(x)$ tend uniformly to $F(x)$ in $[0, M]$ as N tends to infinity. Thus, the inequality (12) will be proved if we show that

$$\int_{-1}^{1} \Phi_N(|f'(x)|)\, dx \leqslant \int_{-1}^{1} \Phi_N(|T_n'(x)|)\, dx \tag{13}$$

for each $f \in \Omega_n$ and every natural number N. But

$$\int_{-1}^{1} \Phi_N(|f'(x)|)\, dx = \sum_{k=1}^{N-1} \beta_k \int_{-1}^{1} \varphi_{\alpha_k}(|f'(x)|)\, dx \tag{14}$$

and the coefficients β_k are positive. Therefore, in order to prove (13), it suffices to show that

$$\int_{-1}^{1} \varphi_\alpha(|f'(x)|)\,dx \leqslant \int_{-1}^{1} \varphi_\alpha(|T_n'(x)|)\,dx \tag{15}$$

for each $\alpha \in (0, M)$ and $f \in \Omega_n$. Further, it follows from the definition of $\varphi_\alpha(x)$ that

$$\int_{-1}^{1} \varphi_\alpha(|f'(x)|)\,dx = \int_{E(\alpha;f)} |f'(x)|\,dx,$$

where $E(\alpha;f) := \{x \in [-1, 1] : |f'(x)| \geqslant \alpha\}$. Clearly $E(\alpha;f)$ consists of non-overlapping intervals. Suppose that $[a, b]$ is one of these intervals. Since $\alpha > 0, f(x)$ is a monotone function in $[a, b]$ and consequently

$$\int_a^b |f'(x)|\,dx = \left| \int_a^b f'(x)\,dx \right| = |f(b) - f(a)|.$$

Suppose that $[a, b] \in I_k$. Let a^* and b^* be the points from the corresponding interval I_k^* for which $T_n(a^*) = f(a)$ and $T_n(b^*) = f(b)$. According to Lemma 2,

$$|T_n'(x)| > \min_{x \in [a, b]} |f'(x)| = \alpha$$

for each $x \in [a^*, b^*]$. Therefore $[a^*, b^*] \subset E(\alpha; T_n)$ and

$$\begin{aligned} \int_{a^*}^{b^*} |T_n'(x)|\,dx &= |T_n(b^*) - T_n(a^*)| \\ &= |f(b) - f(a)| = \int_a^b |f'(x)|\,dx. \end{aligned}$$

Then

$$\int_{E(\alpha;T_n)} |T_n'(x)|\,dx > \int_{E(\alpha;f)} |f'(x)|\,dx$$

and (15) follows. The inequality (12) is proven.

It remains to show that T_n is the unique extremal element in Ω_n. To this end, observe that

$$V(f; [-1, 1]) = \int_{-1}^{1} \varphi_0(|f'(x)|)\,dx.$$

Since $V(T_n; [-1, 1]) = 2n$, there exists an $\varepsilon > 0$ such that

$$\int_{-1}^{1} \varphi_\alpha(|T_n'(x)|)\,dx \geqslant 2n-1$$

for every $\alpha \in [0, \varepsilon]$. Now, assume that $f \neq T_n$. Then $V(f; [-1, 1]) \leqslant 2(n-1)$ (remember that $f \in \Omega_n$) and consequently

$$\begin{aligned}\int_{-1}^{1} \varphi_\alpha(|f'(x)|)\,dx &< V(f; [-1, 1]) \\ &\leqslant -1 + \int_{-1}^{1} \varphi_\alpha(|T_n'(x)|)\,dx\end{aligned} \tag{16}$$

for $\alpha \in [0, \varepsilon]$. Since $0 < \frac{1}{2}F(\varepsilon) < \sum_{\alpha_k \leqslant \varepsilon} \beta_k$ for sufficiently large N, it follows from (14) and (16) that

$$\int_{-1}^{1} \Phi_N(|f'(x)|)\,dx \leqslant -\tfrac{1}{2}F(\varepsilon) + \int_{-1}^{1} \Phi_N(|T_n'(x)|)\,dx$$

which yields (12) with strict inequality, as a limit case.

3. Main Result

We prove in this section the central theorem of the present paper.

THEOREM 1. *Let n be an arbitrary natural number and let $p \in (1, \infty)$. Then*

$$\|f'\|_p \leqslant \|T_n'\|_p \|f\|_C \tag{17}$$

for every polynomial $f \in \pi_n$. Moreover, the equality is attained if and only if $f = \pm T_n$.

Proof. Let the number p be fixed in $(1, \infty)$. Suppose that $f \in \pi_n$, $\|f\|_C = 1$ and

$$\|f'\|_p = \sup\{\|g'\|_p : g \in \pi_n, \|g\|_C \leqslant 1\}.$$

Without loss of generality we assume that $f(\infty) = \infty$. The theorem will be proved if we show that $f = T_n$. Denote by $\{x_k\}_1^{m-1}$, $-1 < x_1 < \cdots < x_{m-1} < 1$, the distinct real zeros of $f'(x)$ in $(-1, 1)$. Evidently $m \leqslant n$. Set, for convenience, $\omega(x) = f'(x)$, $x_0 = -1$, $x_m = 1$. We shall show first that

$$f(x_k) = (-1)^{m-k}, \qquad k = 0, \ldots, m. \tag{18}$$

We investigate the change of the quantities $\|f' + \varepsilon g_k'\|_p$ and $\|f + \varepsilon g_k\|_C$ for small ε, where

$$g_k(x) = (x^2 - 1)\,\omega(x)/(x - x_k).$$

Introduce the function

$$\sigma_k(\varepsilon) := \int_{-1}^{1} |f'(x) + \varepsilon g_k'(x)|^p \, dx.$$

Clearly

$$\sigma_k'(0) = p \int_{-1}^{1} |\omega(x)|^{p-2}\,\omega(x)\,g_k'(x)\,dx. \tag{19}$$

Our first task is to show that

$$\sigma_k'(0) > 0, \qquad k = 0,\ldots, m. \tag{20}$$

In the case $k = 0$ we have

$$\begin{aligned}
\sigma_0'(0) &= p \int_{-1}^{1} |\omega(x)|^{p-2}\,\omega(x)\{(x-1)\,\omega(x)\}'\,dx \\
&= p \int_{-1}^{1} |\omega(x)|^{p-2}\,\omega(x)\{\omega(x) + (x-1)\,\omega'(x)\}\,dx \\
&= p \int_{-1}^{1} |\omega(x)|^p\,dx + \int_{-1}^{1} (x-1)\,d|\omega(x)|^p \\
&= (p-1) \int_{-1}^{1} |\omega(x)|^p\,dx + 2\,|\omega(-1)|^p > 0.
\end{aligned}$$

Similarly one proves that $\sigma_m'(0) > 0$. Now suppose that $0 < k < m$. It is clear from (19) that $\sigma_k'(0) < \infty$ because the integrand is a continuous function in $[-1, 1]$. Then

$$\sigma_k'(0) = \lim_{\delta \to 0} \mathfrak{T}(\delta), \tag{21}$$

where

$$\mathfrak{T}(\delta) = p \int_{\Omega(\delta)} |\omega(x)|^{p-2}\,\omega(x)\,g_k'(x)\,dx$$

and $\quad \Omega(\delta) := [x_0 + \delta, x_1 - \delta] \cup [x_1 + \delta, x_2 - \delta] \cup \cdots \cup [x_{m-1} + \delta, x_m - \delta],$

$\delta > 0$. Let us transform the expression $\mathfrak{T}(\delta)$. After an integration by parts we get

$$\mathfrak{T}(\delta) = A(\delta) - p \int_{\Omega(\delta)} g_k(x)\, d\{|\omega(x)|^{p-2}\,\omega(x)\},$$

where

$$A(\delta) = p \sum_{i=0}^{m-1} \{g_k(x)|\omega(x)|^{p-2}\,\omega(x)|_{x_i+\delta}^{x_{i+1}-\delta}\}.$$

It is easily seen that

$$\lim_{\delta \to 0} A(\delta) = 0. \tag{22}$$

Further,

$$\begin{aligned}
\mathfrak{T}(\delta) &= A(\delta) - p \int_{\Omega(\delta)} \frac{(x^2-1)\,\omega(x)}{x - x_k} (p-1)|\omega(x)|^{p-2}\,\omega'(x)\, dx \\
&= A(\delta) - (p-1) \int_{\Omega(\delta)} \frac{x^2-1}{x-x_k}\, d\,|\omega(x)|^p \\
&= A(\delta) - (p-1) \sum_{i=0}^{m-1} \left\{ \frac{x^2-1}{x-x_k} |\omega(x)|^p \,\bigg|_{x_i+\delta}^{x_{i+1}-\delta} \right\} \\
&\quad + (p-1) \int_{\Omega(\delta)} |\omega(x)|^p \left\{ 1 + \frac{1-x_k^2}{(x-x_k)^2} \right\} dx,
\end{aligned}$$

Now taking into account (21) and (22) we get (20) as a limit case.

We observe that there exist a number $\varepsilon_0 > 0$ and a constant $c > 0$ such that

$$\|f' + \varepsilon g_k'\|_p \geqslant \|f'\|_p + c\varepsilon \tag{23}$$

for every $\varepsilon \in [0, \varepsilon_0]$. This follows immediately from the inequality (20) and the Taylor expansion with respect to ε of the function $\int_{-1}^{1} |f'(x) + \varepsilon g_k'(x)|^p\, dx$. Now let us assume that (18) is not true. Then there exists an $x_k \in \{x_0, ..., x_m\}$ such that $|f(x_k)| < 1$. Therefore $|f(x) + \varepsilon g_k(x)| < 1$ for each x from a neighborhood of x_k, provided ε is sufficiently small. So, in order to estimate the norm $\|f + \varepsilon g_k\|_C$ we have to investigate the function $f(x) + \varepsilon g_k(x)$ near the points x_i, $i \neq k$, only. Since $g_k(x_i) = 0$ for $i \neq k$, it is not difficult to verify that

$$\|f + \varepsilon g_k\|_C = \|f\|_C + \varepsilon\delta(\varepsilon) \tag{24}$$

with some function $\delta(\varepsilon)$ which tends to zero as $\varepsilon \to 0$. Consider the polynomial

$$\psi_\epsilon(x) = [f(x) + \varepsilon g_k(x)]/\|f + \varepsilon g_k\|_C.$$

Obviously $\|\psi_\epsilon\|_C = 1$. In addition, it follows from (23) and (24) that

$$\begin{aligned}\|\psi_\epsilon'\|_p &\geqslant [\|f'\|_p + c\varepsilon]/[1 + \varepsilon\delta(\varepsilon)]\\ &= \|f'\|_p + \varepsilon[c - \delta(\varepsilon)\|f'\|_p]/[1 + \varepsilon\delta(\varepsilon)]\\ &> \|f'\|_p\end{aligned}$$

for sufficiently small positive ε. This contradicts the assumption that f is an extremal element. Therefore $|f(x_k)| = 1$ for each $k = 0,\dots, m$ and our claim (18) follows from the choice of the points $x_1,\dots, x_{m-1}$ as all distinct zeros of $f'(x)$ in $(-1, 1)$. Observe that f is a monotone function between two successive points x_k and x_{k+1}, $k = 0,\dots, m - 1$. Therefore $f \in \Omega_n$. So, we proved that if f is an extremal polynomial, then f must belong to Ω_n. It remains to note that the function $F(x) = |x|^p$ is strictly increasing and convex in $[0, \infty)$ for $0 < p < \infty$ and $F(0) = 0$. The proof is completed by applying Lemma 3.

It is very likely that

$$\|f^{(k)}\|_p \leqslant \|T_n^{(k)}\|_p \|f\|_C$$

for each $f \in \pi_n$, $1 \leqslant p \leqslant \infty$ and $k \in \{0,\dots, n\}$. In any case the conjecture is true for $k = n$, $1 \leqslant p \leqslant \infty$ and for $k \in \{0,\dots, n\}, p = \infty$.

References

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